

Anomalous metapopulation dynamics on scale-free networks

Sergei Fedotov* and Helena Stage†

School of Mathematics, The University of Manchester, Manchester M13 9PL, UK

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We model transport of individuals across a heterogeneous scale-free network where a few weakly connected nodes exhibit heavy-tailed residence times. Such power laws are consistent with the Axiom of Cumulative Inertia, an empirical law stating that the rate at which people leave a place decreases with the associated residence time. We show numerically and analytically that ‘cumulative inertia’ overpowers highly connected nodes in attracting network individuals. Our result, confirmed by empirical evidence, challenges the classical view that individuals will accumulate in highly connected nodes.

Introduction. In the past few decades, many metapopulation models have been developed describing reaction-transport processes on scale-free networks [1–10]. The idea that the overall population can be understood as a series of spatially connected but separated ‘patches’ [11] is useful in many areas including the migration of humans between cities [12], scientific collaborations [13], the spread of epidemic diseases via individual movement [4, 8, 9, 14] and international air travel [15]. Often networks are assumed to be scale-free, such that the order (number of connections) of each node (patch) is drawn from a power law distribution $P(k) \sim k^{-\gamma}$, $\gamma > 0$ [16–19].

While considerations of stochastic movement of individuals on a complex network are very challenging, much progress has been made using a mean-field approximation across nodes of equal order. One introduces the mean number of individuals $N_k(t) = \frac{1}{\eta_k} \sum_i \rho_{i,k}(t)$, where $\rho_{i,k}(t)$ is the number of individuals in the i^{th} node of order k , and η_k is the number of nodes of order k [3–5, 7]. The equation describing transport between nodes can be written as

$$\frac{\partial N_k}{\partial t} = -\mathbb{I}_k(t) + k \sum_{k'} P(k'|k) \frac{\mathbb{I}_{k'}(t)}{k'}, \quad (1)$$

where $\mathbb{I}_k(t)$ is the flux out of a node (patch) of order k and $P(k'|k)$ is the probability that a link exists from a node of order k' to a node of order k [3, 9, 18]. Commonly it is assumed that the residence time in a node (before moving elsewhere) is exponentially distributed [3, 4, 6]. This implies a constant escape rate λ for which the flux is

$$\mathbb{I}_k(t) = \lambda N_k(t), \quad (2)$$

[6, 7, 20]. The assumption of an uncorrelated network, such that $P(k'|k) = \frac{k' P(k')}{\langle k \rangle}$ [18, 21, 22], together with Eq. (2) leads to the well-known steady-state result [3, 4]

$$N_k^s = \frac{k}{\langle k \rangle} \sum_{k'} P(k') N_{k'}^s = \frac{k}{\langle k \rangle} \langle N^s \rangle. \quad (3)$$

It follows from Eq. (3) that the mean number of individuals in a node (patch) increases with the order. One

can interpret this as individuals spending more time in well-connected nodes. This famous result has been key in developing e.g. the Page Rank algorithm and is still fundamental in our intuition regarding network behaviour. However, such conclusions are heavily based on the assumption that the movement between patches can be approximated by a Poisson process. That is, the interval between consecutive escapes from a node (*residence time*), follows an exponential probability density function (PDF) $\psi(\tau) = \lambda e^{-\lambda\tau}$. New work has emerged in recent years indicating that human activity is not Poisson distributed [23]. In particular, the efforts of Barabási and others have demonstrated that human activity often involves heavy-tailed or Pareto type PDFs [19, 24–30]. This is particularly relevant for human mobility due to the empirical sociological law known as ‘*The Axiom of Cumulative Inertia*’ (ACI), which suggests that the probability of a person remaining in a state increases with the associated residence time [31, 32]. The ACI can be reformulated in terms of a power law residence time [33] with PDF:

$$\psi(\tau) = \frac{\mu}{\tau + \tau_0} \left(\frac{\tau_0}{\tau + \tau_0} \right)^\mu \quad (4)$$

for fixed constants $\mu, \tau_0 > 0$. To the authors’ knowledge no work has yet been done investigating the effect of heavy-tailed PDFs like Eq. (4) on Eq. (3), and the subsequent implications for the long-time distribution of network individuals.

So, what happens if we introduce power law PDFs like Eq. (4) into heterogeneous network models? Surprisingly, in the case of $\mu < 1$, the well-known result of Eq. (3) was radically altered beyond the effects attributable to small perturbations. Accumulation in high-order nodes did occur, but as a short-lived transient state of the network. In the long-time limit individuals aggregated in the patches with power-law residence times, invalidating Eq. (3). This *fundamentally challenges the classically held belief* that individuals will tend to accumulate in the nodes of highest order [3, 4, 6–9]. In what follows we develop an anomalous metapopulation model describing this behaviour.

Anomalous Nodes in a Network. We concern ourselves with transport on a heterogeneous scale-free network containing some nodes with power law distributed residence times (see Eq. (4)), and the rest with exponentially distributed waiting times. We call nodes ‘anomalous’ if their average residence time $\langle T \rangle = \int_0^\infty \tau \psi(\tau) d\tau$ diverges. This occurs when $\mu < 1$ and is the case we shall focus on (empirical evidence for its existence to follow). We intend to show that even in the extreme case of few connections, these power law nodes are dominant in attracting network individuals. Individuals leave nodes with rates \mathbb{T} . For exponential residence times, \mathbb{T} is constant and Eq. (2) describes the flux. Else for power law residence times, $\mathbb{T}(\tau) = \frac{\mu}{\tau + \tau_0}$ yields Eq. (4) [33] using $\psi(\tau) = \mathbb{T}(\tau) \exp[-\int_0^\tau \mathbb{T}(u) du]$. We define the survival probability of staying in the node as $\Psi(\tau) = \int_\tau^\infty \psi(u) du$ [34]. The inverse residence time dependence of $\mathbb{T}(\tau)$ is another manifestation of the ACI, which we motivate as follows. Consider a person moving to a new city: over time they acquire a stable existence, develop a social circle, gain steady employment or perhaps enter family life. In other words, the longer they live in the city the more settled they become and are thus less likely to leave [35]. In addition to factors keeping one somewhere, there are further risks associated with moving elsewhere. The more cautious might therefore choose to remain despite potential gain to be had elsewhere [36].

For power law residence times it is convenient to consider the renewal measure $h(t)$. This function can be understood as the number of events per unit time, where an ‘event’ is an individual leaving a node. $h(t)$ obeys the renewal equation

$$h(t) = \psi(t) + \int_0^t h(\tau) \psi(t - \tau) d\tau \quad (5)$$

[34]. One can rewrite the flux $\mathbb{I}_k(t)$ from Eq. (1) as

$$\mathbb{I}_k(t) = \frac{d}{dt} \int_0^t h(t - \tau) N_k(\tau) d\tau, \quad (6)$$

which is valid for all $\psi(\tau)$ (see [37], Ch. 5 for the derivation). Clearly, $h(t) = \lambda$ yields Eq. (2). Similarly, $\mu > 1$ in Eq. (4) allows for a well-defined mean residence time $\langle T \rangle$ which in the long time limit produces a flux analogous to Eq. (2) where $\lambda \approx \frac{1}{\langle T \rangle} = \frac{\mu - 1}{\tau_0}$. A critical case arises when $\mu < 1$ and this is no longer possible. Then, the renewal measure follows

$$h(t) = \frac{t^{-1+\mu}}{\Gamma(1 - \mu)\Gamma(\mu)\tau_0^\mu} \quad (7)$$

as $t \rightarrow \infty$ [37, 38]. Substituting Eq. (7) into Eq. (6), we obtain the anomalous flux $\mathbb{I}_k^a(t)$

$$\mathbb{I}_k^a(t) = \frac{1}{\Gamma(1 - \mu)\tau_0^\mu} {}_0\mathcal{D}^{1-\mu} N_k(t), \quad \mu < 1 \quad (8)$$

where ${}_0\mathcal{D}^{1-\mu}$ is the Riemann-Liouville operator [37, 39–41] (details in Supplementary Information). This flux appropriately describes the notion of ‘cumulative inertia’.

We will show that Eq. (8) changes the preferential residence of individuals in well-connected nodes in favour of those with anomalous flux, even if these are weakly connected. This corresponds to dominance of low-order nodes (patches) with flux \mathbb{I}_k^a over high-order nodes with flux $\mathbb{I}_k = \lambda N_k(t)$. Let us for simplicity assume only anomalous nodes to have order $k_a \ll \langle k \rangle$ (nodes are weakly connected). The flux $\mathbb{I}_k(t)$ from the balance Eq. (1) becomes

$$\mathbb{I}_k(t) = [1 - \delta_{kk_a}] \lambda N_k(t) + \delta_{kk_a} \mathbb{I}_k^a(t), \quad (9)$$

where δ_{kk_a} is the discrete Kronecker delta. By analysis of Eq. (1) (details in Supplementary Information), it follows that in the limit $t \rightarrow \infty$

$$N_k(t) \eta_k \rightarrow \delta_{kk_a} N, \quad (10)$$

where N is the total number of individuals in the network, and η_k the number of nodes with order k . Hence the anomalous nodes jointly contain all individuals as $t \rightarrow \infty$. This key result contrasts with the popular belief that well-connected nodes are more attractive. Furthermore, similar results cannot be replicated by naïvely introducing nodes with very low escape rates $\lambda \ll 1$.

We confirm the result of Eq. (10) by Monte Carlo simulations illustrated in Figure 1. A scale-free ($P(k) \sim k^{-\gamma}$, $\gamma = 1.5, 2.5$), uncorrelated network was constructed using the Molloy-Reed algorithm, containing $\eta_{k_a} = 3$ anomalous nodes of order $k_a = 4$ [42]. This was compared with another network where all nodes have exponential residence times and flux $\mathbb{I}_k(t) = \lambda N_k(t)$. Both simulations were carried out with 100 nodes and $N = 10^5$ individuals.

The simulation almost immediately showed the individuals accumulating in nodes according to their order as described by Eq. (3). However, this behaviour was transient as the individuals then slowly moved into the anomalous nodes. We observed an initially fast rate of organisation into the classically expected configuration which then, with a (power law) slow rate, changed into a preference for the anomalous nodes. This leads to the peak in $N_k(t)$ at $k = k_a$. As the network is scale-free, it may also contain other nodes of order k_a which do not have anomalous flux (and thus are gradually emptied). This does not qualitatively alter our results, but reduces the value of $N_{k_a}(t)$ as η_{k_a} grows.

Some work has already been done considering the effects of a flux like Eq. (8) on single states [43–45]. However, the result of such dynamics when pitched against the attraction of well-connected nodes has not previously been developed. This difference is essential for networks designed to represent heavy-tailed processes. Related work exists considering heavy-tailed residence times in

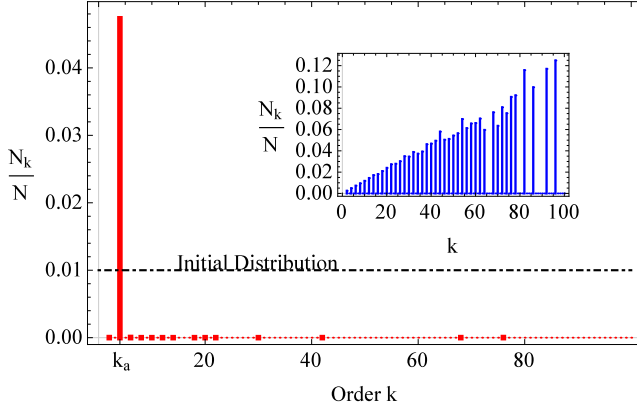


FIG. 1. $\frac{N_k(t)}{N}$ for a network of 100 nodes with 3 anomalous nodes, all of order $k_a = 4$ with $\mu = 0.5$, $\tau_0 = 1$, and $N = 10^5$ individuals (initially distributed uniformly). The individuals eventually aggregate in the anomalous nodes. The inset shows $\frac{N_k}{N}$ if all nodes have constant escape rates $T = \lambda = 2$.

biased Watts-Strogatz networks, which demonstrated pair aggregation akin to self-chemotactic-like forcing [40]. Other pattern formation on scale-free networks has been observed with order-dependent escape rates [46]. Indeed, patterns or dominant behaviours are known to arise in networks, either as a result of heterogeneities in $P(k)$ [20] and the role of extreme values of k [47], or following the interplay of these with escape rates or node reaction dynamics [5].

Two-State System. From our simulations we observe the formation of two states in the network. There is a slow transport of individuals to the anomalous patches arising from the gradual depletion of the surrounding nodes. Consequently, we can regard this peak in individuals as one state S_1 and the remainder of the nodes as the other state S_2 . This picture (see Figure 2) allows us to find the rate at which the aforementioned peak grows. The corresponding equations to Eq. (1) are

$$\frac{dN_1}{dt} = \mathbb{I}_2(t) - \mathbb{I}_1(t), \quad N_2(t) = N - N_1(t) \quad (11)$$

where $N_i(t)$, $\mathbb{I}_i(t)$ are the respective mean number of individuals in, and flux from, state S_i . Hence the fluxes $S_2 \leftrightarrow S_1$ in analogy to Eq. (2) and Eq. (6) are given by $\mathbb{I}_2(t) = \lambda N_2(t)$, and $\mathbb{I}_1(t) = \frac{d}{dt} \int_0^t h(t-\tau) N_1(\tau) d\tau$ where $h(t)$ follows Eq. (7). In the limit of $t \rightarrow \infty$ we neglect the derivative $dN_1/dt \approx 0$ such that Eq. (11) becomes

$$N = N_1(t) + \frac{1}{\lambda} \frac{d}{dt} \int_0^t h(t-\tau) N_1(\tau) d\tau. \quad (12)$$

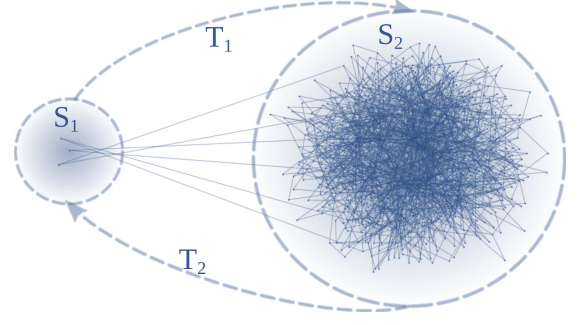


FIG. 2. Network separation into two states 1, 2 with transition rates T_1 , T_2 . The exact number of nodes in each state and the number of connections between the states is insignificant, so long as S_2 contains the majority of nodes. The intention is to demonstrate the attractiveness of S_1 , even in the extreme case where there are very few connections.

This evaluates to

$$N_1(t) = N \left(1 - \frac{h(t)}{\lambda} \right) \rightarrow N, \quad N_2(t) = \frac{Nh(t)}{\lambda} \rightarrow 0. \quad (13)$$

as $t \rightarrow \infty$. Eq. (13) thus describes the power law slow, non-stationary aggregation which is consistent with Eq. (10). This phenomenon has been observed previously in other contexts [48], though its implications for networks has hitherto not been considered. Internal connections in S_2 are negligible as they simply contribute slightly to the probability of remaining in S_2 (thus increasing the time taken to aggregate in S_1 , but not the overall behaviour).

Using the same parameters, Monte Carlo simulations of the whole network were carried out to test the prediction of Eq. (13) and the validity of the two-state simplification. As seen in the inset of Figure 3, the simulation is in agreement with theoretical expectations and shows the convergence to Eq. (13) in the long time limit. The suitability of the fit thus supports our two-state simplifying assumption. We note that even at large times (small) oscillations occur around the maximum, indicating that an equilibrium state does not exist.

Preferential Residence. The aim of this subsection is to provide empirical evidence for the anomalous attractiveness of nodes with power law residence time PDF like Eq. (4) with $\mu < 1$. Eq. (13) and Figure 1 show that individuals will tend to reside in S_1 , but what is the fine structure of the residence time in the state? We separate the number of individuals according to their residence times. Hence $n_1(t, \tau) \Delta \tau$ gives the number of individuals with residence times in the interval $(\tau, \tau + \Delta \tau)$ with initial condition $n_1(0, \tau) = n_1^0 \delta(\tau)$ where $n_1^0 \ll N$. Consequently, $N_1(t) = \int_0^t n_1(t, \tau) d\tau$. We can write n_1 in terms of the renewal measure $h(t)$ [34]

$$n_1(t, \tau) = Nh(t-\tau)\Psi(\tau), \quad (14)$$

where the survival function $\Psi(\tau) = \int_\tau^\infty \psi(u) du$ can be

obtained from Eq. (4) to yield $\Psi(\tau) = \left(\frac{\tau_0}{\tau + \tau_0}\right)^\mu$. Substituting Eq. (7) and letting $t \rightarrow \infty$, we find

$$n_1(t, \tau) \simeq \frac{N}{\Gamma(1 - \mu)\Gamma(\mu)\tau^\mu (t - \tau)^{1-\mu}}. \quad (15)$$

This result is consistent with the generalised arc sine distributions for backwards recurrence times [49] (see p.445 where $x = \tau/t$), which we remind the reader is only valid for $\mu < 1$. Our aim now is to demonstrate that Eq. (15) is consistent with empirical observations. By analysing data from an objective housing survey carried out among 16000 households in Milwaukee between 1950-1962, we obtained the residence times since moving into the current home [50]. This was done over an interval of 12 years and allows us to ‘track’ households and their moves as illustrated in Figure 3. The key features of the plot are the peaks in $\frac{n_1}{N}$ at $\tau \ll t$ and $\tau \sim t$, corresponding to the most likely residence times being very short or constituting the majority of the time. The same behaviour is produced by Eq. (15), and is qualitatively very different from the predictions for $\mu > 1$. In the latter case where the mean residence time $\langle T \rangle$ exists, it is well-known that one obtains the asymptotic result $n_1(t, \tau) \rightarrow \frac{\Psi(\tau)}{\langle T \rangle}$ [34]. This is a decaying function of residence time τ and does not provide a good description of the data in Figure 3.

The presence of peaks at both low and high residence times in our data agrees with the findings of the Axiom of Cumulative Inertia, in that most of the individuals with either be long-term residents (which do not move), or the sum of the continued in/outflux of new arrivals. Our findings are consistent with similar data obtained by the Bureau of Census during the American Housing Surveys in the period 1985-1993 [51].

A crucial point to note here is that the data suggests *human residence falls into the case of anomalous behaviour*. Despite our analysis only being valid in cases $\mu < 1$, it is seen that this is a ubiquitous example in population movement, with variations arising depending on the nature of residence (renting/owning a home). We therefore expect our model to be generally representative of human residence, and consequently of significance for any number of networks wishing to model residence dynamics akin to human movement.

In our model we have assumed from the beginning that some nodes are anomalous. However, the aggregation can be made self-organising (enabling it to occur spontaneously) by modifying the form of the escape rate to $\mathbb{T}_k(\tau, N_k) = \frac{\mu(N_k)}{\tau + \tau_0}$, where $\mu(N_k)$ is a decreasing function of the mean number of individuals. This negative dependence has previously been observed in small mammals and gulls [52, 53], and is treated theoretically in [10, 45, 54]. One could then have $\mu(N_k(t)) > 1$ until a sufficiently large number of individuals were present in a node to reduce its value to $\mu(N_k(t)) < 1$. This time-

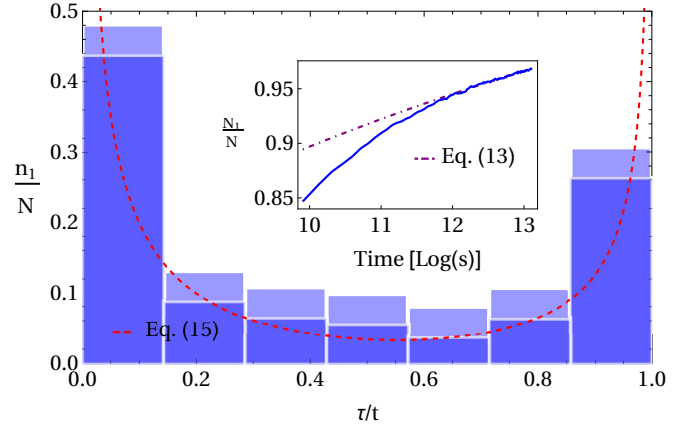


FIG. 3. The histogram shows $\frac{n_1(t, \tau)}{N}$, sampled from 12288 households in Milwaukee from 1950-1962. There is reasonable agreement between the data and Eq. (15) for $\mathbb{T}(\tau) \approx 0.55/(0.22 + \tau)$ at $t = 12$ years between 1950-62. Estimated errors are indicated by the shaded regions. The inset shows $\frac{N_1(t)}{N}$ as measured from our simulations (using same parameters as Figure 1), thus illustrating the aggregation of individuals in \mathbb{S}_1 as described by Eq. (13).

dependent evolution in the value of μ can be interpreted as a phase transition in the distribution of individuals across the network, where the mean residence time $\langle T \rangle$ in a node would transition from finite to infinite. While not essential in producing the effects of Eq. (13), the properties inherent to $\mathbb{T}_k(\tau, N_k)$ settle any concerns that the aggregation behaviour is forced from the outset. So long as $\mu(N_1) > 1$ is it possible to find the steady-state mean number of individuals N_1^s, N_2^s :

$$N_1^s = \lambda N_2^s \int_0^\infty \left(\frac{\tau_0}{\tau_0 + \tau}\right)^{\mu(N_1^s)} d\tau = \lambda N_2^s \langle T^s \rangle, \quad (16)$$

where $\langle T^s \rangle = \int_0^\infty \Psi(\tau, N_1^s) d\tau$ is finite (for details see [45]). When $\mu(N_1) < 1$ the above solution no longer exists (loses physical meaning) and Eq. (16) thus constitutes a criterion for whether a phase transition takes place. Above the threshold $\mu(N_1) = 1$ we have

$$N_1^s = \frac{N\lambda\tau_0}{\mu(N_1^s) - 1 + \lambda\tau_0}, \quad N_2^s = N - N_1^s. \quad (17)$$

Discussion and Conclusion. It is a commonly held belief that individuals in a scale-free network will prefer highly connected nodes (patches). Our work fundamentally challenges this notion when individuals’ flux follows the Axiom of Cumulative Inertia as described by Eq. (8). In this anomalous case, we have shown both analytically and numerically that the flux out of nodes with power law residence times outperforms highly connected nodes in the aggregation of network individuals. We further provide empirical evidence for the physical relevance of this particular case ($\mu < 1$), motivated by the ACI [36].

There is an initial, fast convergence to the classically expected distribution of individuals according to the node orders, but this is found to be a transient state. In other words, the slow non-Markovian dynamics in the system dominate over the ‘fast’ Markovian ones. Our findings constitute an important result in the context of network theory given the wealth of evidence that human behaviour, such as our habits on web surfing and with television, follows heavy-tailed distributions [28–30]. Other examples of such distributions include messaging, queuing and prioritising tasks [25, 27, 55].

Our analysis and simulations have been carried out in the extreme case of the anomalous nodes having very few connections. However, our results are valid for all $k_a > 0$; the number of connections simply affects how quickly the accumulation of individuals will occur. An example of such a network is the movement of workers between cities and the surrounding towns. Some cities, either by repute or growing economy, may attract workers with promises of higher wages, better job security, etc [31, 32]. These would be highly ordered anomalous nodes. Other cities, perhaps as a result dwindling demand for goods/services offered or geographical factors (like climate change or local resource scarcity), would also be in need of workers but not necessarily capable of offering the same conditions. These are nodes with a high number of connections but a steady escape rate, as there is little keeping the workers from seeking ‘greener pastures’ [36].

An open question is the impact of our findings on networks containing reaction-transport processes, such as those inherent to epidemiology (e.g. the SIR model) [4, 8, 9, 14]. The effects of residence time dependencies in epidemiological invasion thresholds have been considered in [56]. We expect anomalous patches to be significant in understanding how diseases might spread when individuals are reluctant to leave an area. This will be the subject of future work.

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* sergei.fedotov@manchester.ac.uk

† helena.stage@manchester.ac.uk

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SUPPLEMENTARY INFORMATION

Asymptotic Results

Let us consider Eq. (6) in more detail. If a node of order k has escape rate $\mathbb{T}_k(\tau) = \frac{\mu}{\tau + \tau_0}$, we can write an intuitive description of the flux to be

$$\mathbb{I}_k(t) = \int_0^t \mathbb{T}_k(\tau) n_k(t, \tau) d\tau = \int_0^t \frac{\mu}{\tau + \tau_0} n_k(t, \tau) d\tau, \quad (\text{I})$$

where $n_k(t, \tau)$ is the structured density of individuals. That is, $n_k(t, \tau)\Delta t$ gives the number of individuals in a node of order k with a residence times in the interval $(\tau, \tau + \Delta\tau)$. It thus follows that $N_k(t) = \int_0^t n_k(t, \tau) d\tau$ and so for a constant rate \mathbb{T}_k we obtain Eq. (2). The structured density obeys the equation of motion

$$\frac{\partial n_k(t, \tau)}{\partial t} + \frac{\partial n_k(t, \tau)}{\partial \tau} = -\mathbb{T}_k(\tau) n_k(t, \tau), \quad (\text{II})$$

which we can solve using the method of characteristics to obtain $n_k(t, \tau) = n_k(t - \tau, 0) e^{-\int_0^\tau \mathbb{T}_k(u) du}$. Here, $n_k(t - \tau, 0)$ is the number of new arrivals in the node of order k from elsewhere. We define the exponential term to be the survival function such that $n_k(t, \tau) = n_k(t - \tau, 0) \Psi_k(\tau)$. It follows from the definition that $\psi_k(\tau) = -\frac{\partial \Psi_k}{\partial \tau} = \mathbb{T}_k(\tau) \Psi_k(\tau)$. By integration we find

$$N_k(t) = \int_0^t n_k(t - \tau, 0) \Psi_k(\tau) d\tau, \quad (\text{III})$$

which by substitution into (I) gives $\mathbb{I}_k(t) = \int_0^t n_k(t - \tau, 0) \psi_k(\tau) d\tau$. By application of the Laplace transform ($\mathcal{L}_t\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt = \hat{f}(s)$ denotes the Laplace transformation of $f(t)$) we find

$$\hat{\mathbb{I}}_k(s) = \hat{n}_k(s, 0) \hat{\psi}_k(s) = \frac{\hat{\psi}_k(s)}{\hat{\Psi}_k(s)} \hat{N}_k(s) = s \hat{h}_k(s) \hat{N}_k(s), \quad (\text{IV})$$

where we have used (III) and Eq. (4) in the last two steps. Consequently, by an inverse Laplace transformation $\mathbb{I}_k(t) = \frac{d}{dt} \int_0^t h_k(\tau) N_k(t - \tau) d\tau$ and we obtain Eq. (6).

In the long-time limit of $t \rightarrow \infty$ (or equivalently $s \rightarrow 0$ in Laplace space) we can find the the Laplace transform $\hat{\psi}(s)$ of the PDF given in Eq. (4). One finds $\hat{\psi}(s) = [1 + (s\tau_0)^\mu \Gamma(1 - \mu)]^{-1}$, and so the renewal measure obeys $\hat{h}(s) = \frac{\hat{\psi}(s)}{1 - \hat{\psi}(s)} = [(s\tau_0)^\mu \Gamma(1 - \mu)]^{-1}$. By an inverse Laplace transformation we obtain Eq. (7). Using the definition of the Riemann-Liouville operator, which for $0 < \mu < 1$ has the form:

$${}_0\mathcal{D}^{1-\mu} N_k(t) = \frac{d}{dt} \int_0^t N_k(t - \tau) \frac{\tau^{\mu-1}}{\Gamma(\mu)} d\tau \rightarrow \mathcal{L}_t\{{}_0\mathcal{D}^{1-\mu} N_k(t)\}(s) = s^{1-\mu} \hat{N}_k(s), \quad (\text{V})$$

we can (using (IV) and $\hat{h}(s)$) express the flux in terms of this quantity in the asymptotic limit. So

$$\mathbb{I}_k^a(t) = \frac{{}_0\mathcal{D}^{1-\mu}N_k(t)}{\Gamma(1-\mu)\tau_0^\mu}, \quad (\text{VI})$$

as $t \rightarrow \infty$, which is consistent with Eq. (8).

Identifying nodes as either anomalous or not, we can substitute Eq. (9) into Eq. (1) along with the assumption of an uncorrelated network $P(k'|k) = \frac{k'P(k')}{\langle k \rangle}$ to yield

$$\frac{1}{\lambda} \frac{\partial N_k}{\partial t} = \delta_{kk_a} \left[\frac{k}{\langle k \rangle} \sum_{k' \neq k} P(k') N_{k'} - \frac{\mathbb{I}_k^a}{\lambda} \right] + (1 - \delta_{kk_a}) \left[\frac{k}{\langle k \rangle} \left(P(k_a) \frac{\mathbb{I}_{k_a}^a}{\lambda} + \sum_{k' \neq k, k_a} P(k') N_{k'} \right) - N_k \right]. \quad (\text{VII})$$

Transforming (VII) into Laplace space and letting $s \rightarrow 0$ (equivalent to the long-time limit $t \rightarrow \infty$) we can compare the relative values of the terms to find that $\frac{\partial N_k}{\partial t} \approx 0$. Similarly, we find the dominant behaviour $\sum_{k' \neq k_a} P(k') N_{k'}(t) \gg \mathbb{I}_{k_a}^a(t)$ and $\mathbb{I}_{k_a}^a(t) \ll \sum_{k' \neq k, k_a} P(k') N_{k'}(t)$. Setting these terms to zero, we obtain

$$0 \approx \delta_{kk_a} \sum_{k' \neq k} P(k') N_{k'} + (1 - \delta_{kk_a}) \left[\sum_{k' \neq k, k_a} P(k') N_{k'} - \frac{\langle k \rangle}{k} N_k \right]. \quad (\text{VIII})$$

When $k = k_a$, the mean number of individuals outside the anomalous node $\sum_{k' \neq k_a} P(k') N_{k'} = 0$, and so the entire population must be present in the anomalous nodes. This leads to the total aggregation of individuals in nodes of order k_a as described by Eq. (10).

Two-State Simplification

The purpose of this section is to show that we can qualitatively approximate the long-time behaviour of the network into two states. The intention is not to prove that the overall equations exactly reduce to Eq. (11). Consider two states in (VII): $k = k_a$ and $k \neq k_a$ (which we shall term Ω). Hence we get

$$\frac{\partial N_{k_a}}{\partial t} = \frac{k_a}{\langle k \rangle} \lambda \sum_{k' \neq k_a} P(k') N_{k'} - \mathbb{I}_{k_a}^a(t) = \frac{k_a}{\langle k \rangle} \lambda \langle N_{k'}(t) \rangle_{k' \neq k_a} - \mathbb{I}_{k_a}^a(t), \quad (\text{IX})$$

where $\frac{k_a}{\langle k \rangle} \lambda \langle N_{k'}(t) \rangle_{k' \neq k_a}$ represents the average influx from other nodes into the anomalous nodes. This approximates \mathbb{S}_1 . Similarly, the non-anomalous nodes follow

$$\sum_{k \neq k_a} \frac{\partial N_k}{\partial t} = \frac{\partial N_\Omega}{\partial t} = \sum_{k \neq k_a} \frac{k}{\langle k \rangle} \left(P(k_a) \mathbb{I}_{k_a}^a(t) + \lambda \langle N_{k'}(t) \rangle_{k' \neq k, k_a} \right) - \lambda N_\Omega. \quad (\text{X})$$

$\sum_{k \neq k_a} \frac{k}{\langle k \rangle} P(k_a) \mathbb{I}_{k_a}^a(t)$ is the average anomalous flux into all the other nodes, and $\sum_{k \neq k_a} \frac{k}{\langle k \rangle} \lambda \langle N_{k'}(t) \rangle_{k' \neq k, k_a} - \lambda N_\Omega$ represents all connections in/out of order $k \neq k_a$. This approximates \mathbb{S}_2 . As we consider most all these nodes as the state Ω , these are ‘internal’ movements in the state and thus cancel out, with the exception of any connections from nodes of order $k \rightarrow k_a$. The result is a scaling in the value of λN_Ω and letting $\sum_{k \neq k_a} \frac{k}{\langle k \rangle} \lambda \langle N_{k'}(t) \rangle_{k' \neq k, k_a} \approx 0$, which leads to the qualitatively similar Eq. (11).